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Projection of Orbits and K Multiplicities

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In this article we give an explicit formula for the push forward of the Liouville measure of regular coadjoint orbits of connected, noncompact, semisimple Lie groups under the moment map arising from the Hamiltonian action of a maximal compact subgroup. We also discuss its connection with Frobenius reciprocity.

1. INTRODUCTION

The aim of this article is the explicit computation of the push forward under proper moment maps (for definitions, etc., see [6]) of the Liouville measure of a class of K Hamiltonian spaces where K is a compact connected Lie group. More specifically our setting is the following (for unexplained notations see the next section).

Let G be a connected, noncompact, semisimple Lie group with finite centre and let $\mathfrak{G}$ be its Lie algebra. We choose a Cartan decomposition $\mathfrak{G} = \mathbf{k} + \mathbf{p}$ of $\mathfrak{G}$ and denote by K (resp. θ) the maximal compact subgroup of G with Lie algebra $\mathbf{k}$ (resp. Cartan involution) of G. We will denote also by θ, the differential of θ at e, the identity element of G. Let $\mathfrak{h}$ be a θ stable (nonsplit) Cartan subalgebra of $\mathfrak{G}$. Then we have $\mathfrak{h} = \mathbf{t}_0 + \mathfrak{a}_0$ where $\mathbf{t}_0 = \mathfrak{h} \cap \mathbf{k}$ and $\mathfrak{a}_0 = \mathfrak{h} \cap \mathbf{p}$. We denote the Killing form of $\mathfrak{G}$ by B, recall that this is a nondegenerate, G invariant, symmetric, bilinear form on $\mathfrak{G}$ and hence this provides us with a G equivariant identification of $\mathfrak{G}$ with $\mathfrak{G}^*$. Furthermore the restriction of B to $\mathbf{k}$ (resp. $\mathbf{p}$) is negative definite (resp. positive definite). We identify $\mathfrak{h}^*$ with the annihilator in $\mathfrak{G}^*$ of the subspace $[\mathfrak{h}, \mathfrak{G}]$ of $\mathfrak{G}$ and for $\lambda \in \mathfrak{h}^*$ we write $\lambda = \lambda_0 + \lambda_1$ with $\lambda_0 \in \mathbf{t}_0^*$ and $\lambda_1 \in \mathfrak{a}_0^*$ via the (dual) decomposition $\mathfrak{h}^* = \mathbf{t}_0^* + \mathfrak{a}_0^*$. Let $\lambda \in \mathfrak{h}^*$ be $\mathfrak{G}$ regular and $\Omega = G \cdot \lambda \subset \mathfrak{G}^*$ its coadjoint orbit. Ω is a G (hence a fortiori K) Hamiltonian space. The moment map for the Hamiltonian action of K on Ω which we denote by J is the restriction to Ω of the orthogonal projection from $\mathfrak{G}^*$ onto $\mathbf{k}^*$. J is a K equivariant proper map. Hence the push

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forward under J of the Liouville measure, β_Ω, of Ω is a K invariant measure on $\mathbf{k}^*$. We denote this measure by $J_*\beta_\Omega$. Our aim in the sequel is to give an explicit formula for $J_*\beta_\Omega$.

The paper is organised as follows. Section 2 introduces notations, Section 3 gives a motivation for studying the problem at hand, and Sections 4 through 7 give the statement and proof of the formula for $J_*\beta_\Omega$. The final section (8) is concerned with a heuristic "interpretation" of the formula for $J_*\beta_\Omega$.

The result in this paper is a generalization of a theorem of Duflo *et al.* [1]. They computed $J_*\beta_\Omega$ when Ω is a regular, elliptic coadjoint orbit.

2. Notations

If G is a Lie group, we denote its (real) Lie algebra by $\mathfrak{G}$ and by G^0 its identity component. Analogous notation is used for other groups (but we have also employed corresponding lower case Latin (script) letters to denote Lie algebras). If V is a finite-dimensional vector space (real or complex) V^* denotes its dual. Moreover if V is real, $V^{\mathbb{C}}$ will denote its complexification. Fourier transforms of tempered distributions are denoted by a superscript $\wedge$. If ϕ is a function on V we denote by $\check{\phi}$ the function given by $\check{\phi}(v) = \phi(-v)$, $v \in V$. This notation will also be extended to distributions on V. We denote restrictions of functions (distributions) to subsets of their domain by a vertical bar. Finally the symbol i denotes a fixed square root of -1.

3. Connection with the K-Character of Representations of G

$J_*\beta_\Omega$ is a K invariant tempered measure on $\mathbf{k}^*$. Its Fourier transform is therefore a K invariant tempered distribution on $\mathbf{k}$. We want to discuss here the connection between $J_*\beta_\Omega$ and the K-character of irreducible unitary representations of G. This is a consequence of Kirillov's character formula which we state below. Let $j^{1/2}(X) = \det^{1/2}(\sinh \operatorname{ad}(X/2)/\operatorname{ad}(X/2))$, $X \in \mathfrak{G}$. Kirillov's character formula says that to every irreducible tempered representation π of G with regular infinitesimal character one can attach a regular orbit $\Omega(\pi)$ of G in $\mathfrak{G}^*$. Without loss of generality we can assume that such a $\Omega(\pi)$ is of the form $G \cdot \lambda$ for a suitable regular $\lambda \in \mathfrak{h}^*$ where $\mathfrak{h}$ is a θ stable Cartan subalgebra of $\mathfrak{G}$. This correspondence has the property that if Θ_π denotes the character of π then one has

$$\Theta_\pi(\exp X) = j^{-1/2}\hat{\beta}_{\Omega(\pi)} \tag{1}$$

The above is an equality of distributions in a suitable G invariant neighbourhood (call it V, say) of the origin in $\mathfrak{G}$.

We denote the centre of the universal enveloping algebra, $U(\mathfrak{G}^{\mathbb{C}})$, by $z(\mathfrak{G}^{\mathbb{C}})$ and the ring of G invariant constant coefficient differential operators on $\mathfrak{G}$ by $I(\mathfrak{G}^{\mathbb{C}})$. Every G invariant distribution on G (resp. $\mathfrak{G}$) which is $z(\mathfrak{G}^{\mathbb{C}})$ finite (resp. $I(\mathfrak{G}^{\mathbb{C}})$ finite) can be restricted to the group K (resp. $\mathbf{k}$). Since the distribution Θ_π (resp. $\hat{\beta}_\Omega$) has these properties [Appendix [1]], we have in $V \cap \mathbf{k}$

$$j_{\mathbf{p}}^{1/2}(X)\, \Theta_\pi|_K(\exp X) = j_{\mathbf{k}}^{-1/2}(X)\, \hat{\beta}_\Omega|_{\mathbf{k}}(X), \tag{2}$$

the symbols $j_{\mathbf{k}}^{1/2}$ (resp. $j_{\mathbf{p}}^{1/2}$) having their obvious meanings. Now

$$\Theta_\pi|_K = \sum_{i \in \hat{K}} m_i \pi_i \tag{3}$$

where π_i denotes the character of the element $i \in \hat{K}$, the unitary dual of K and m_i is the multiplicity of i in the restriction of π to K. A classical result of Harish-Chandra [3] says that m_i is at most equal to the degree of the representation i. Kirillov's character formula (applied to the compact connected group K) yields

$$\pi_i(\exp Y) = j_{\mathbf{k}}^{-1/2}(Y)\, \hat{\beta}_{K \cdot \nu_i}, \tag{4}$$

where $\nu_i \in \mathbf{k}^*$ is a $\mathbf{k}$ regular element. Equality (4) is an equality of analytic functions on $\mathbf{k}$ in this case. We therefore on $V \cap \mathbf{k}$,

$$\hat{\beta}_\Omega|_{\mathbf{k}} = j_{\mathbf{p}}^{1/2} \sum_{i \in \hat{K}} m_i \hat{\beta}_{K \cdot \nu_i} \tag{5}$$

Thus $\hat{\beta}_\Omega|_{\mathbf{k}}$ is (locally) completely determined by the multiplicities m_i and [Appendix [1]] $\hat{\beta}_\Omega|_{\mathbf{k}}$ is the Fourier transform of $J_* \beta_\Omega$. We remark here that although Eq. (5) is purely local and makes sense only for those orbits Ω which correspond to irreducible tempered representations, such as π above, nevertheless the formula for $J_* \beta_\Omega$ that we obtain appears to reflect an "analytic continuation" to all regular orbits the essence of (5).

4. The Formula for $J_* \beta_\Omega$

In this section we will give the formula for $J_* \beta_\Omega$. This necessitates the introduction of several notations which are explained below. We regard $\mathbf{p}^{\mathbb{C}}$ as a $\mathbf{t}_0^{\mathbb{C}}$ module under the adjoint action. We employ the symbols Δ, Δ_{SIm}, Δ_{CIm}, Δ_{Cpx} for the set of all roots of the pair $(\mathfrak{G}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$, singular imaginary roots, compact imaginary roots, complex roots, respectively. Let Q be the set of nonzero weights of $\mathbf{t}_0^{\mathbf{C}}$ in $\mathbf{p}^{\mathbf{C}}$. It is clear that $Q = \Delta_{\mathrm{SIm}} \cup \{\alpha|_{\mathbf{t}_0^{\mathbf{C}}} \mid \alpha \in \Delta_{\mathrm{Cpx}}\}$. Now $\Delta(\mathfrak{G}^{\mathbf{C}}, \mathbf{t}_0^{\mathbf{C}}) = Q \cup \Delta(\mathbf{k}^{\mathbf{C}}, \mathbf{t}_0^{\mathbf{C}})$ where $\Delta(\mathfrak{G}^{\mathbf{C}}, \mathbf{t}_0^{\mathbf{C}})$ (resp. $\Delta(\mathbf{k}^{\mathbf{C}}, \mathbf{t}_0^{\mathbf{C}})$) denotes the set of nonzero weights of $\mathbf{t}_0^{\mathbf{C}}$ in $\mathfrak{G}^{\mathbf{C}}$ (resp. $\mathbf{k}^{\mathbf{C}}$). Let L (resp. l) denote the centralizer of $\mathfrak{a}_0$ in G (resp. $\mathfrak{G}$). We have $L = MA_0$ (resp. $l = \mathfrak{M} + \mathfrak{a}_0$) where $A_0 = \exp \mathfrak{a}_0$. L and M are θ stable reductive subgroups of G with $L \cap K = M \cap K$ and $\operatorname{rank} M = \operatorname{rank} M \cap K$. Let $\mathbf{k}_0 = \mathfrak{M} \cap \mathbf{k}$, and let r be the orthocomplement of $\mathbf{k}_0$ in $\mathbf{k}$ with respect to B. Then we have $\Delta(\mathbf{k}^{\mathbf{C}}, \mathbf{t}_0^{\mathbf{C}}) = \Delta(\mathbf{k}_0^{\mathbf{C}}, \mathbf{t}_0^{\mathbf{C}}) \cup \Delta(r^{\mathbf{C}}, \mathbf{t}_0^{\mathbf{C}})$. Now, by definition, $\Delta(\mathbf{k}_0^{\mathbf{C}}, \mathbf{t}_0^{\mathbf{C}}) = \Delta_{\mathrm{CIm}}$ and $\Delta(r^{\mathbf{C}}, \mathbf{t}_0^{\mathbf{C}}) = \{\alpha|_{\mathbf{t}_0^{\mathbf{C}}} \mid \alpha \in \Delta_{\mathrm{Cpx}}\}$. λ is $\mathfrak{G}$ regular implies that λ_0 is $\mathfrak{M}$ regular and $\Delta(\mathfrak{M}^{\mathbf{C}}, \mathbf{t}_0^{\mathbf{C}}) = \Delta_{\mathrm{Im}} = \Delta_{\mathrm{SIm}} \cup \Delta_{\mathrm{CIm}}$. Hence λ_0 determines a unique system of positive imaginary roots which we denote by $\Delta_{\mathrm{Im}}^+(\lambda_0) = \{\alpha \in \Delta_{\mathrm{Im}} \mid (i\lambda_0, h_\alpha) > 0\}$ where h_α is the coroot corresponding to α. A choice of order in $\Delta(\mathfrak{G}^{\mathbf{C}}, \mathbf{t}_0^{\mathbf{C}})$ compatible with $\Delta_{\mathrm{Im}}^+(\lambda_0)$ hence gives an order in Q consistent with $\Delta_{\mathrm{SIm}}^+(\lambda_0) = \Delta_{\mathrm{Im}}^+(\lambda_0) \cap \Delta_{\mathrm{SIm}}$. We denote the resulting set of positive elements in $\Delta(\mathfrak{G}^{\mathbf{C}}, \mathbf{t}_0^{\mathbf{C}})$ (resp. Q) by $\Delta^+(\mathfrak{G}^{\mathbf{C}}, \mathbf{t}_0^{\mathbf{C}})$ (resp. Q^+). We enumerate the elements of Q^+ as $\{\mu_1, \ldots, \mu_m\}$, say.

Let z denote the centralizer of $\mathbf{t}_0$ in $\mathbf{k}$. We clearly have $\Delta(\mathbf{k}^{\mathbf{C}}, \mathbf{t}_0^{\mathbf{C}}) = \Delta(\mathbf{k}^{\mathbf{C}}, \mathbf{t}^{\mathbf{C}}) \backslash \Delta(z^{\mathbf{C}}, \mathbf{t}^{\mathbf{C}})$ (restriction to $\mathbf{t}_0^{\mathbf{C}}$ being understood). A choice of a system of positive roots $\Delta^+(z^{\mathbf{C}}, \mathbf{t}^{\mathbf{C}})$ along with $\Delta^+(\mathfrak{G}^{\mathbf{C}}, t_0^{\mathbf{C}})$ hence furnishes us with a positive system, $\Delta^+(\mathbf{k}^{\mathbf{C}}, \mathbf{t}^{\mathbf{C}})$. We will for brevity write Δ_z^+ in place of $\Delta^+(z^{\mathbf{C}}, \mathbf{t}^{\mathbf{C}})$ and let π_z stand for the product of the elements of Δ_z^+. We will denote also (by abuse of notation) by π_z, the product $(i)^{\# \Delta_z^+} \prod_{\alpha \in \Delta_z^+} h_\alpha$. Letting $\mathbf{t}_1$ be the orthocomplement of $\mathbf{t}_0$ in $\mathbf{t}$ with respect to B, π_z can be viewed as a polynomial function on $\mathbf{t}_1^*$.

Let $\mathbf{k} = \mathbf{t} + \tilde{r}$ where $\tilde{r}$ is the orthocomplement of $\mathbf{t}$ in $\mathbf{k}$. The dimension of $\tilde{r}$ is even, call it $2n$. If $\eta \in \mathbf{t}^*$ let K_η be the two form on $\tilde{r}$ defined by $K_\eta(X, Y) = \langle \eta, [X, Y] \rangle$. If η is $\mathbf{k}$ regular then the form K_η is nondegenerate. We choose a nonvanishing form μ of degree $2n$ on $\tilde{r}$. We define a polynomial function π on $\mathbf{t}^*$ by the formula

$$K_\eta = \pi(\eta)\, n!\, \mu,$$

where π is a W anti-invariant polynomial function on $\mathbf{t}^*$ depending on μ. Here W is the Weyl group of T in K. π is proportional to $\pi_C^+ = \prod_{\alpha \in \Delta^+(\mathbf{k}^{\mathbf{C}}, \mathbf{t}^{\mathbf{C}})} h_\alpha$. We choose μ so that $\pi(\eta)$ is a positive multiple of $\pi_C^+(i\eta) \eta \in \mathbf{t}^*$ and denote the resulting π by π^+.

We choose Lebesgue measures dk (resp. dt) on $\mathbf{k}$ (resp. $\mathbf{t}$) so that $dk = dt \cdot |\mu|$. Whenever we choose a Lebesgue measure on the Lie algebra of a Lie group we will use the left invariant Haar measure on the group so that $d(\exp X)/dX$ is equal to 1 for $X = 0$. We denote by vol T the integral $\int_T dT$. We now define a map A^+ from $C_c(\mathbf{k}^*)$ to $C_c(\mathbf{t}^*)$ by the prescription

$$A^+\phi(\eta) = \frac{1}{\# W(\operatorname{vol} T)} \cdot \frac{\pi^+(\eta)}{(2\pi)^n} \int_K \phi(k \cdot \eta)\, dK. \tag{6}$$

In an identification of $\mathbf{k}$ with $\mathbf{k}^*$, by a K invariant scalar product on $\mathbf{k}$, A^+ reduces essentially to Harish-Chandra's invariant integral [4].

THEOREM. $\forall \phi \in C_c(\mathbf{k}^*)$, *we have,*

$$\langle J_* \beta_\Omega, \phi \rangle = C \left\langle \sum_{w \in W} \varepsilon(w) w \cdot (\mathbf{H}_{i\mu_1} * \cdots * \mathbf{H}_{i\mu_m} * \delta_{\lambda_0} \otimes \pi_z), A^+ \phi \right\rangle.$$

Here H denotes the Heaviside distribution and C is a constant which may depend on the Cartan subalgebra $\mathfrak{h}$ but does not depend on $\lambda \in \mathfrak{h}^*$.

5. SOME OBSERVATIONS

We first make some preliminary observations before giving the proof of the theorem. We will identify $\mathfrak{G}$ with $\mathfrak{G}^*$ via B and let $X = X_0 + X_1 \in \mathfrak{h}$ be the element corresponding to $\lambda = \lambda_0 + \lambda_1$. There is a Euclidean structure on $\mathfrak{G}$, denoted $(\ ,\ )$ which is given by $(X, Y) = -B(X, \theta Y)$ for $X, Y \in \mathfrak{G}$. We denote the corresponding norm on $\mathfrak{G}$ by $\|\ \|$. We identify Ω with $G \cdot X$ and assume that $X_0 \neq 0$.

PROPOSITION 1. $J(\Omega) = \{k \in \mathbf{k} \mid \|k\| \geq \|X_0\|\}$. *Furthermore* $J^{-1}(K \cdot X_0) = K \cdot X$, *the K orbit of X in Ω.*

Proof. We have $\mathfrak{G} = l + n^+ + \theta n^+$, an orthogonal decomposition with respect to $(\ ,\ )$ where

$$(n^+)^{\mathbb{C}} = \bigoplus_{\substack{\alpha \in \Delta^+(\mathfrak{G}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}) \\ \alpha \mid \mathfrak{a}_0 \neq 0}} \mathfrak{G}_\alpha, \mathfrak{G}_\alpha$$

the root subspace of $\mathfrak{G}^{\mathbb{C}}$ corresponding to the root α. Let $N = \exp n^+$. The map $f: K \times M^0 \times A_0 \times N \to G$, given by $f(k, m, a_0, n) = kma_0n$ for $k \in K$, $m \in M^0$, $a \in A_0$, $n \in N$, is surjective. In view of the K equivariance of J we get $J(g \cdot X) = k \cdot J(ma_0n \cdot X)$. The mapping $n \to n \cdot X - X$ is an isomorphism of N with n^+ (since X is regular). Both M and A_0 normalize N. Let P denote the orthogonal projection onto $\mathbf{k}$. Then $P: n^+ \to r$ is an isomorphism: if $\{z_j\}_{j=1}^s$ is an orthonormal basis of n^+ (with respect to $(\ ,\ )$) than $\{\sqrt{2}\, Pz_j\}_{j=1}^s$ is an orthonormal basis of r. Since $\mathbf{k}_0$ and r are orthogonal with respect to $(\ ,\ )$, we get by K invariance of $(\ ,\ )|_{\mathbf{k} \times \mathbf{k}}$ that $\|J(y)\| \geq \|J(m \cdot X_0)\|$ for $y \in \Omega$, $m \in M^0$. Clearly $\|J(m \cdot X_0)\| \geq \|X_0\|$. Hence $\|J(y)\| \geq \|X_0\| > 0$.

Let $y = kman \cdot X \in \Omega$ with $J(y) \in K \cdot X_0$, the K orbit of X_0 in $\mathbf{k}$. It follows from the above discussion that we must have $n = e$, $J(y) \in K \cdot X_0 \Rightarrow m \in K_0 = M^0 \cap K$. Therefore $y \in K \cdot X$.

6. Proof of the Theorem

Let $\psi \in C_c(\mathbf{k}^*)$. We have the following formula [7]:

$$\langle J_* \beta_\Omega, \psi \rangle = \text{constant} \int_K \int_{M^0} \int_{n^+} J^* \psi(k \cdot (mX + Z))\, dK\, dm\, dZ. \tag{7}$$

Here dK = normalized Haar measure of K, i.e., $\int_K dK = 1$; dm = standard Haar measure of M^0 (for a definition see [7]); dZ = Euclidean measure on n^+ derived from (,).

In the following, for brevity, we will drop constants of proportionality whenever convenient. We will show later on that the constant C in the theorem is independent of λ. With this understood, the right-hand side of Eq. (7) becomes, with dy = Euclidean measure on r and ψK invariant,

$$\int_{M^0} \int_r \psi(J(M^0 \cdot X_0) + y)\, dm\, dy \qquad (\text{since } P: n^+ \to \mathbf{r} \text{ is an isomorphism})$$

$$= C(X_0) \langle J_* \beta_{M^0 \cdot X_0} \otimes dy, \psi \rangle \tag{7'}$$

Here $C(X_0)$ is a constant depending on X_0. $J_* \beta_\Omega$ is therefore proportional to the average $\int_K k \cdot (J_* \beta_{M^0 \cdot X_0} \otimes dy)\, dK$. We will compute this average below by using an approximation technique which is rather natural in this circumstance.

Let $\phi_0 \in C_c^\infty(\mathbf{t}_0)$ and let D_0 be the K_0 invariant distribution on $\mathbf{k}_0$ (which is a function in this case) given by $\langle D_0, \phi \rangle = \langle d_0, A_0^+ \phi \rangle$ with $d_0 = \sum_{w_0 \in W_0} \varepsilon(w_0)\, w_0 \cdot \phi_0$. Here W_0 is the Weyl group of $\Delta(\mathbf{k}_0^{\mathbb{C}}, \mathbf{t}_0^{\mathbb{C}}) = \Delta_{\mathbb{C}\mathrm{Im}}$, $\phi \in C_c(\mathbf{k}_0)$, the symbol A_0^+ having its obvious meaning (note that X_0 is $\mathbf{k}_0$ regular). Let D be the distribution on $\mathbf{k}$ given by $\langle D, f \rangle = \int_K \langle D_0 \otimes dy, k \cdot f \rangle$, $f \in C_c^\infty(\mathbf{k})$. D is obviously a K invariant distribution on $\mathbf{k}$.

Lemma 1. *$\langle D, f \rangle = \text{constant} \langle d^+, A^+ f \rangle$ with*

$$d^+ = \sum_{w \in W} \varepsilon(w) w \cdot (\phi_0 * H_{-i\mu_1} * \cdots * I_{-i\mu_l} \otimes \pi_z)$$

and

$$\{\mu_1, ..., \mu_l\} = \Delta^+(r^{\mathbb{C}}, \mathbf{t}_0^{\mathbb{C}}) \subset Q^+.$$

Proof. Let $P_0: \mathbf{k} \to \mathbf{k}_0$ be the orthogonal and let $\psi \in C_c^\infty(\mathbf{k})$ be a K invariant function. Then by the definition of D we have $\langle D, \psi \rangle = \int_{\mathbf{k}} \psi(X)\, D_0(P_0(X))\, dX$ and

$$\int \psi(X)\, D_0(P_0(X))\, dX = \text{constant} \int_{\mathbf{t}} \pi^+(H)^2\, \psi(H) \int_K D_0(P_0(k \cdot H))\, dK\, dH \tag{8}$$

(here we are considering π^+ as a function on $\mathbf{t}$ rather than $\mathbf{t}^*$).

Now $P_0\colon K \cdot H \to \mathbf{k}_0$ is the moment map for the Hamiltonian action of K_0 on the K orbit $K \cdot H$. This is the key idea for proving the lemma. We denote this moment map by J_0. Now $|\pi^+(H)| \int_K D_0(J_0(k \cdot H))\, dK$ is proportional to $\langle (J_0)_* \beta_{K \cdot H}, D_0 \rangle$. We define a W_0 invariant distribution, q_H, on $\mathbf{t}_0$ by $\langle (J_0)_* \beta_{K \cdot H}, f \rangle = \langle q_H, A_0^+ f \rangle$, $f \in C_0^\infty(\mathbf{k}_0)$. Therefore $\langle (J_0)_* \beta_{K \cdot H}, D_0 \rangle = \langle q_H, A_0^+ D_0 \rangle = \langle q_H, d_0 \rangle = \# W_0 \langle q_H, \phi_0 \rangle$ since q_H and d_0 areboth W_0 anti-invariant. Since $\pi^+(H)^2 = \pi^+(H) \operatorname{sign} \pi^+(H)$ $|\pi^+(H)|$, the function we have to evaluate is $H \to \operatorname{sign} \pi^+(H) \langle q_H, \phi_0 \rangle$.

We write $H = H_0 + H_1$ corresponding to the orthogonal decomposition $t = \mathbf{t}_0 + \mathbf{t}_1$ ($H_0 \in \mathbf{t}_0$, $H_1 \in \mathbf{t}_1$) where H is any $\mathbf{k}$ regular element in $\mathbf{t}$. Let W_z denote the Weyl group of the pair $(z^{\mathbb{C}}, \mathbf{t}^{\mathbb{C}})$, $\rho_z = \frac{1}{2} \sum_{\alpha \in \Delta_z^+} \alpha$. We have the following equality of distributions on $\mathbf{t}_0$.

PROPOSITION 2 [5].

$$\operatorname{sign} \pi^+(H) q_H = \text{constant} \cdot \left(\sum_{w \in W} \varepsilon(w)\, \pi_z(w \cdot H)\, \delta_{(w \cdot H)_0} \right) * H_{i\mu_1} * \cdots * H_{i\mu_l} \tag{9}$$

In the above we regard $\{\mu_1, ..., \mu_l\}$ (*cf. Lemma* 1) *as a subset of* $i\mathbf{t}_0$ *rather than* $i\mathbf{t}_0^*$.

We therefore get that $H \to \operatorname{sign} \pi^+(H) \langle q_H, \phi_0 \rangle$ is proportional to $\langle (\sum_{w \in W} \varepsilon(w)\, \pi_z(w \cdot H)\, \delta_{(w \cdot H)_0}) * H_{i\mu_1} * \cdots * H_{i\mu_l}, \phi_0 \rangle$. The latter can be written as $\sum_{w \in W} \varepsilon(w)\, \pi_z((w \cdot H)_1)(\phi_0 * H_{i\mu_1} * \cdots * H_{i\mu_l})((w \cdot H)_0)$. Hence we obtain $d^+ = \sum_{w \in W} \varepsilon(w)\, w \cdot (\phi_0 * H_{-i\mu_1} * \cdots * H_{-i\mu_l} \otimes \pi_z)$. Q.E.D.

We now let ϕ_0 be a distribution with compact support on $\mathbf{t}_0$, $d_0 = \sum_{w_0 \in W_0} \varepsilon(w_0)\, w_0 \cdot \phi_0$, and D_0 the K_0 invariant distribution on $\mathbf{k}_0$ corresponding to d_0. Let $D = \int_K k \cdot (D_0 \otimes dy)\, dK$, d^+ the corresponding W anti-invariant distribution on $\mathbf{t}$. Let $\{f_n\}$ be a delta sequence in $\mathbf{t}_0$, i.e., $f_n \in C_c^\infty(\mathbf{t}_0)$ and $f_n \to \delta$ as $n \to \infty$ in the weak topology of distributions on $\mathbf{t}_0$. Let $\phi_0^n = \phi_0 * f_n \in C_c^\infty(\mathbf{t}_0)$ and let $d_0^n = \sum_{w_0 \in W_0} \varepsilon(w_0)\, w_0 \cdot \phi_0^n$. The map $\phi_0 \to d^+$ is continuous in the weak topology and hence by continuity d_n^+, the W anti-invariant distribution on $\mathbf{t}$ corresponding to d_0^n, converges to d^+, i.e., we have

$$d^+ = \sum_{w \in W} \varepsilon(w) w \cdot (H_{-i\mu_1} * \cdots * H_{-i\mu_l} * \phi_0 \otimes \pi_z). \tag{10}$$

Now let S denote the following subspace of $\mathscr{D}'(\mathbf{t}_0)$, the space of distributions on $\mathbf{t}_0$:

$$S = \{T \in \mathscr{D}'(\mathbf{t}_0) \mid \text{support of } H_{-i\mu_1} * \cdots * H_{-i\mu_l} \cap [C\text{-support of } T] \text{ is compact, where } C \text{ is any compact subset of } \mathbf{t}_0\}.$$

$\mathscr{E}'(\mathbf{t}_0)$, i.e., the distributions in $\mathbf{t}_0$ having compact support, are dense in S. For any $T \in S$, $H_{-i\mu_1} * \cdots * H_{-i\mu_l} * T$ is a well-defined distribution on $\mathbf{t}_0$. Furthermore if a sequence $\{T_n\}$ with $T_n \in S\ \forall n$ converges to $T \in S$ then $H_{-i\mu_1} * \cdots * H_{-i\mu_l} * T_n$ converges to $H_{-i\mu_1} * \cdots * H_{-i\mu_l} * T$. Hence the sequence d_n^+ where $d_n^+ = \sum \varepsilon(w) w \cdot (H_{-i\mu_1} * \cdots * H_{-i\mu_l} * T_n \otimes \pi_z)$ converges to the distribution $d^+ = \sum_{w \in W} \varepsilon(w)\, w \cdot (H_{-i\mu_1} * \cdots * H_{-i\mu_l} * T \otimes \pi_z)$. Now $(A^+)^t : \mathscr{D}'(\mathbf{t})^{W\text{anti-invariant}} \to \mathscr{D}'(\mathbf{k})^{K\text{invariant}}$ is a continuous map (in fact a topological isomorphism) and hence $(A^+)^t d_n^+ = D_n$ (say) converges to $(A^+)^t d^+ = D$. We therefore have that Eq. (10) remains valid for any K_0 invariant distribution, D_0, on $\mathbf{k}_0$ (not necessarily having compact support) such that the W_0 anti-invariant distribution on $\mathbf{t}_0$, d_0, corresponding to it is of the form $d_0 = \sum_{w_0 \in W_0} \varepsilon(w_0) w_0 \cdot T$ with $T \in S$. A theorem of Duflo *et al.* [1] gives an explicit formula for $(J_0)_* \beta_{M^0 \cdot \lambda_0}$ which shows that it is of the above form.

THEOREM [1]. $\forall \phi \in C_c(\mathbf{k}_0^*)$

$$\langle J_* \beta_{M^0 \cdot \lambda_0}, \phi \rangle = \sum_{w_0 \in W_0} \varepsilon(w_0)\, w_0 \cdot (\delta_{\lambda_0} * H_{-i\mu_{l+1}} * \cdots * H_{-i\mu_m}), A_0^+ \phi \rangle.$$

Here $\mu_{l+1}, ..., \mu_m = \Delta^+_{\mathrm{SIm}}(\lambda_0) \subset Q^+$. This concludes the proof of our theorem.

7. Some Comments on the Constant C

In this section we indicate why the constant C in the theorem is independent of λ. It suffices to show that the product of the constant in Eq. (7) with $C(X_0)$ is independent to X. The constants which arise in the averaging process occur due to choices of Lebesgue measures and hence are independent of X. We can also assume that the Cartan subgroup H is abelian since otherwise we get merely a multiplicative factor $[H : Z(H)]^{-1}$ where $Z(H) =$ centre of H and this number is clearly independent of X.

Let $d =$ dimension of G, $l = \frac{1}{2}$(dimension $G -$ rank G). We denote by $d_0 X$ (resp. $d_0 H$) the normalized Euclidean measure on $\mathfrak{G}$ (resp. $\mathfrak{h}$) so that if we define the Fourier transform on $\mathfrak{G}$ (resp. $\mathfrak{h}$) by $\hat{f}(Y) = \int f(X)\, e^{iB(X,Y)}\, d_0 X$

$(f \in \mathscr{S}(\mathfrak{G}), X, Y \in \mathfrak{G})$ we have $\hat{\hat{f}}(Y) = \check{f}(Y) = f(-Y)$. The G invariant measure $d_{G/H}$ on G/H can be normalized so that $\forall f \in C_c(G \cdot \mathfrak{h}_{\mathrm{reg}})$ we have

$$\int f(X)\, d_0 X = \int_{G/H} \int_{\mathfrak{h}_{\mathrm{reg}}} |\pi(H)|^2 f(\dot{x}H)\, d_{G/H}(\dot{x})\, d_0 H. \tag{11}$$

Here π is the product of the elements of $\Delta^+(\mathfrak{G}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$ which we choose so that $\Delta^+(\mathfrak{G}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}}) \cap \Delta_{\mathrm{Im}} = \Delta^+_{\mathrm{Im}}(\lambda_0)$.

The Kirillov measure of the orbit $\Omega = G \cdot X$ $(\Omega \approx G/H)$ is given by $|\pi(X)|\ \# W(G, H)(2\pi)^{-l}\, d_{G/H}$ where $W(G, H)$ is the Weyl group of H in G. Proposition 8.1.3.4 of [7] gives

$$\int_{G/H} f(\dot{x}X)\, d_{G/H}(\dot{x})$$

$$= C_G\, |\pi_{R,C}(X)|^{-1} \int_K \int_{M^0} \int_{n^+} f(k \cdot (m \cdot X + Z))\, dK\, dm\, dz.$$

Here the notation is as in Eq. (7) and C_G is a constant depending on G which will be stated explicitly later, and $\pi_{R,C}$ is the product of non-imaginary elements of $\Delta^+(\mathfrak{G}^{\mathbb{C}}, \mathfrak{h}^{\mathbb{C}})$. Thus the constant (call it C', say) in Eq. (7) is

$$|\pi_{\mathrm{Im}}(X)|\ \# W(G, H)(2\pi)^{-l} C_G \qquad \Big(\pi_{\mathrm{Im}} = \prod_{\alpha \in \Delta^+_{\mathrm{Im}}(\lambda_0)} \alpha\Big). \tag{12}$$

Now the Kirillov measure of the orbit $M^0 \cdot X_0$ is given by $\# W_0\, |\pi_{\mathrm{Im}}(X)| \cdot C_{M^0}(2\pi)^{-\# \Delta^+_{\mathrm{Im}}(\lambda_0)}\, dm$. Hence we have $C(X_0) = [\# W_0\, |\pi_{\mathrm{Im}}(X)|\ C_{M^0}(2\pi)^{-\# \Delta^+_{\mathrm{Im}}(\lambda_0)}]^{-1}$ and consequently

$$C'C(X_0) = \frac{\# W(G, H)(2\pi)^{-\# \Delta^+_{R,C}}}{\# W_0 \cdot C_{M^0}}\, C_G. \tag{13}$$

This proves our assertion since both C_G and C_{M^0} are absolute constants (i.e., independent of λ in this context).

We have

$$C_G^{-1} = \# W \prod_{\alpha \in \Delta^+(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})} (\alpha, \rho_C)(2\pi)^{m_G - r_G}(2)^{n_G}, \tag{14}$$

where $m_G = \frac{1}{2}(\dim G/K - \operatorname{rank} G + \operatorname{rank} K)$; $n_G = \frac{1}{2}(\dim G/K - \operatorname{rank} G/K)$; $r_G = \frac{1}{2}(\dim G - \operatorname{rank} G)$; $\rho_C = \frac{1}{2} \sum^{\alpha}_{\alpha \in \Delta^+(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})}$.

Remark 1. In the case $\mathfrak{h}$ is a fundamental Cartan subalgebra of $\mathfrak{G}$ it is possible to give an analytic proof of Enright's multiplicity formula [2] for the K multiplicities of the regular (i.e., having regular infinitesimal character) fundamental series representations of G using our formula for $J_* \beta_\Omega$. The proof is entirely similar (albeit more messy) to the proof of Blattner's formula given in [1] and is therefore omitted.

8. Some Heuristic Considerations

We close our paper with an "interpretation" of the formula for $J_* \beta_\Omega$ that we have obtained as a "continuous" manifestation of the assertion of the Frobenius reciprocity theorem in the representation theory of compact Lie groups. The discussion here is formal; nevertheless we found the ensuing heuristics make the formula for $J_* \beta_\Omega$ appear plausible. It is for this reason we have decided to include it.

We recall (see Section 2) that $J_* \hat{\beta}_\Omega$, which is the restriction to $\mathbf{k}$ of $\hat{\beta}_\Omega$, is locally completely determined by the K-multiplicities in case Ω is an admissible orbit corresponding to an irreducible tempered representation (call it π, say) of G with regular infinitesimal character. According to Harish-Chandra's description of irreducible tempered representations, π is induced from a representation of $P = MAN$ of the form $\delta \otimes v \otimes 1$ where δ is a discrete series representation of M, v is a unitary character of A, and 1 denotes the trivial representation of N. In the above P is a cuspidal parabolic subgroup of G with Langlands decomposition, $P = MAN$. The restriction of π to K is the induced representation $\mathrm{Ind}_{M \cap K}^{K} \delta|_{M \cap K}$. We denote by $m(\mu, \pi)$ the multiplicity of $\mu \in \hat{K}$ in the restriction of π to K and assume for simplicity that M is connected. The Frobenius reciprocity theorem tells us that

$$m(\mu, \pi) = \sum_{v \in (M \hat{\cap} K)} m(v, \delta|_{M \cap K})\, m(v, \mu|_{M \cap K}). \tag{15}$$

The term $m(v, \delta|_{M \cap K})$ is given by Blattner's formula whilst asymptotic estimates of $m(v, \mu|_{M \cap K})$ have been obtained by G. Heckmann [5]. In view of the remark made in the beginning of the previous paragraph we can regard d_0(cf. Section 5) as a "continuous" analogue of $\sum_{v \in (M \hat{\cap} K)} m(v, \delta|_{M \cap K})$ and Heckmann has shown that the distribution q_H (see Section 5) is a "continuous" analogue of $m(v, \mu|_{M \cap K})$ where H corresponds to μ in the parametrization of $\hat{K}$ as regular admissible coadjoint orbits. In fact, under certain assumptions on the set $\Delta^+(r^{\mathbb{C}}, \mathfrak{t}_0^{\mathbb{C}})$ [see [5]], in which case q_H is a continuous piecewise polynomial function on $\mathbf{t}_0$, Heckman has proved that q_H is indeed the multiplicity function $m(v, \mu|_{M \cap K})$ in case the pair $(\mathbf{k}, \mathbf{k}_0)$ is of the type $(su(n+1), u(n))$ or $(so(n+2), so(n+1))$ and suitable v [see Lemma 3.8 of [5]]. Now $(M \hat{\cap} K)$ consists of lattice points lying in a cone. We denote the weight lattice of $M \cap K$ by Γ and the subset of dominant weights by Γ^+. We extend $m(\nu, \delta|_{M\cap K})$ and $m(\nu, \mu|_{M\cap K})$ (a priori defined only for $\nu \in \Gamma^+$) to all of Γ by defining them to be zero if $\nu \notin \Gamma^+$. We will denote these extended functions by the same symbols. If f and g are Z^+ valued functions on Γ, at least one of which has finite support, then their convolution is again a Z^+ valued function on Γ and $\sum_{\gamma\in\Gamma} f(\gamma)\, g(\gamma) = (f * \check{g})(0)$. Applying this to (15) with $f = m(\nu, \delta|_{M\cap K})$ and $g = m(\nu, \mu|_{M\cap K})$ we obtain

$$m(\mu, \pi) = [m(\nu, \delta|_{M\cap K}) * m(\nu, \mu\check{|}_{M\cap K})](0). \tag{16}$$

We continue assuming that q_H is a continuous piecewise polynomial function. Then since q_H has compact support $(d_0 * \check{q}_H)(0)$ makes sense. The proof of the theorem given in Section 6 shows that this is essentially what is taking place. (We saw that the antisymmetrization with respect to W_0 which occurs in d_0 disappears). Thus we can regard our formula for $J_*\beta_\Omega$ as a "continuous" manifestation of Frobenius reciprocity since a general orbit Ω need not correspond to any tempered, irreducible representation of G but the convolution suggested by (16) is present in our formula.

Remark 2. Equation (15) shows that the multiplicities of the K types are independent of the unitary character ν in the inducing representation $\delta \otimes \nu \otimes 1$ of P. The description of $J(\Omega)$ given in Proposition 1 shows that it is independent of $X_1 \in \mathfrak{a}_0$.

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